Assignment III Math 2010 Prof. Y. Kwong

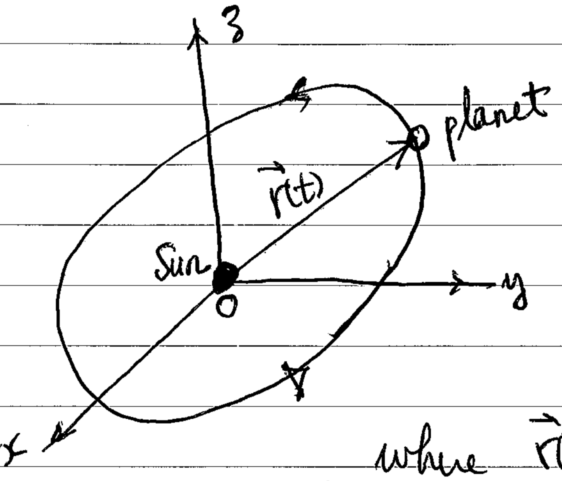
Date P.1

1. Find the length of the following curves:

(i) $x=\frac{t^2}{2}$, $y=\ln t$, $z=\sqrt{2}t$ from $t=1$ to $t=2$

(ii) $x=3t\sin t$, $y=3t\cos t$, $z=2t^2$ from $t=0$ to $t=4/5$.

2. Consider the planetary motion around the Sun. Without loss of generality, we assume the Sun is at the origin of the co-ordinate system.



According to Newton's Law of Universal Gravitation, the planet is attracted towards the Sun by a force

$$\vec{F} = \frac{-GMm\,\vec{r}(t)}{|\vec{r}(t)|^3}$$

where $\vec{r}(t)$ being the position vector from the Sun to the planet at time t. Now that along with Newton's 2nd law of motion,

$\vec{F} = m\vec{a}(t)$ (M & m are respectively the mass of the Sun and mass of planet),

prove that the vector product $\vec{r}(t)\times\vec{v}(t)$ is a constant vector.

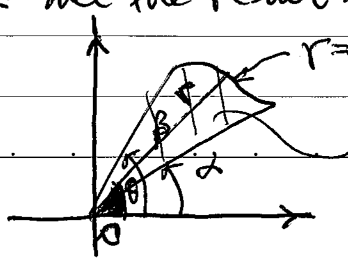
3. Without loss of generality, let us now assume that the orbit of the planet is the xy plane, then we could assume the position vector $\vec{r}(t)$ take the form:

$$\vec{r}(t) = \langle r(t)\cos\theta(t),\ r(t)\sin\theta(t),\ 0\rangle$$

and hence show that $\vec{r}(t)\times\vec{v}(t) = r^2(t)\frac{d\theta}{dt}\vec{k}$ where $r(t) = |\vec{r}(t)|$

4. Using the result of problem 3 to prove Kepler's Law of planetary motion i.e the position vector $\vec{r}(t)$ swepts out equal areas inside the orbit in equal time intervals.

(Hint: use the result that area being swept out by a polar curve is given by



$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$).

Assignment III Prof. Y. Kwong

5/. Let a curve in space be parametrized in cylindrical co-ordinates i.e. $r = r(t)$, $\theta = \theta(t)$, $z = z(t)$ are functions of t for $a \le t \le b$. Prove that the length L of the curve is given by:

$$L = \int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

6/. Write an equation of the plane through the point $(1,1,1)$ that is normal to the twisted curve parametrized as $x = t$, $y = t^2$, $z = t^3$ at the point.

7/. Given $\vec{a}(t) = \langle e^{-3t}, t, \sin t \rangle$ with $\vec{v}(0) = \langle 4, -2, 4 \rangle$ and $\vec{r}(0) = \langle 0, 4, -2 \rangle$, find $\vec{r}(t)$.

8/. Describe and sketch the surface or solid region described by the following equations or inequalities.

(i) $\theta = 3\pi/4$ (ii) $\phi = \frac{5\pi}{6}$ (iii) $z - 4r^2 = 2$ (iv) $z^2 - 2r^2 = 4$

(v) $0 \le \phi \le \pi/6$, $0 \le \rho \le 10$.

9/. Show that $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

10/. Based on the ϵ-δ definition of limit, prove that for $f(x,y) = 3x + 2y$, $\lim\limits_{(x,y)\to(2,1)} f(x,y) = 8$.